Introduction to Set Theory by Stephen Taylor
http://composertools.com/Tools/PCSets/setfinder.html

1. **Pitch Class** The 12 notes of the chromatic scale, independent of octaves. "C" is the same pitch class, no matter whether it's middle C or the top C on the piano keyboard. Thus there are only 12 pitch classes. Enharmonic spellings make no difference, so B#, C, and Dbb are all the same pitch class.

2. **Pitch Class Set** Any collection of pitch classes, e.g. [C, D, Eb, F]. Pitch classes are not repeated in a set. For example, the beginning of "Happy Birthday" is not [C, C, D, C, F, E], but rather [C, D, F, E]. Sets are typically displayed in brackets [ ] or parentheses ( ).

There are two kinds of pc sets, **ordered** and **unordered**. Ordered pc sets are used in serial music; for now, we will work only with unordered pc sets. Ordered sets are more like a melody or motive, while unordered sets are more like chords or scales ("raw material" from which you can build a melody).

3. **Cardinality** The number of pitch classes in a set. [C, E, G] has a cardinality of 3.

4. **Subset** A part of a pitch class set. For example, the subsets of [C, F#, G] are [C, F#], [F#, G], and [C, G], as well as the single-element sets [C], [F#], and [G].

5. **Numeric Representation** By arbitrarily choosing one pc as a reference point (0), we can number all the other pcs in a set. Usually, C = 0, giving us the following numbers for each half-step:

c	c#	d	d#	e	f	f#	g	g#	a	a#	b
0	1	2	3	4	5	6	7	8	9	10	11

Here are some numeric representations of sets which you should recognize:

Major triad	[0, 4, 7]
Minor triad	[0, 3, 7]
Major $^6{}_4$ triad	[0, 5, 9]
Minor $^6{}_3$ triad	[0, 4, 9]
Whole tone scale	[0, 2, 4, 6, 8, 10]

Sets are usually written in ascending order, starting with the lowest number.

6. **Modulo 12** Sometimes called "clock-face arithmetic"; a mathematical concept which limits any number so that it falls within a certain range. On a 12-hour clock, for instance, there are no numbers higher than 12. If we add 3 hours to 11:00, we get 2:00, not 14:00.

Music works the same way. If we add three scale steps to G, we don't get H, I, J; instead we get A, B, C. Below are some examples in numeric representation:

11 + 3 = 2 (B-natural + 3 half steps = D-natural)
2 + 13 = 3 (D-natural + 13 half steps = D#)
1 - 2 = 11 (C# - 2 half steps = B-natural)

7. **Transposition** In set theory, think of transposition as addition or subtraction using mod 12. For example, to transpose [C, E, G, Bb] up a perfect 5th:

1. [C, E, G, Bb] = [0,4,7,10]
2. Add 7 (mod12) to each pc in the set:

$$\begin{array}{ccccc} & 0 & 4 & 7 & 10 \\ + & \underline{7} & \underline{7} & \underline{7} & \underline{7} \\ & 7 & 11 & 2 & 5 \end{array}$$

3. The result is [7, 11, 2, 5], which we write as [2, 5, 7, 11].

We can call this set Transposition 7, or T7, of the old set. “T” always means to transpose *up*. Negative numbers do not need to be used for transpositions (although they can), since downward transposition can always be converted to a positive number (e.g. transposition down a perfect fourth is the same as T7, up a perfect fifth).

8. **Interval Class** You are probably already familiar with the concept of octave inversion of intervals (M3 becomes m6, P5 becomes P4, etc.). Any interval can be reduced to an interval of a tritone or less, by changing one of the pitches’ octaves. Doing this produces an *interval class*. There are only six interval classes, notated with < >:

ic <1> m2 or M7
ic <2> M2 or m7
ic <3> m3 or M6
ic <4> M3 or m6
ic <5> P4 or P5
ic <6> tritone

9. **Interval Content** All of the possible interval classes contained within a pc set. Let’s use the previous set as an example:

[C, E, G, Bb] = [0, 4, 7, 10]

C to E = M3, or ic <4>
C to G = P5, or ic <5>
C to Bb = m7, or ic <2>
E to G = m3, or ic <3>
E to Bb = d5, or ic <6>
G to Bb = m3, or ic <3>

This pc set, with a cardinality of 4, contains no members of ic <1>, one member or ic <2>, two of ic <3>, and one each of ic <4>, ic <5>, and ic <6>.

10. **Interval Vector** A simple way of showing the interval content of a pc set. Interval vectors are displayed with < >, just like interval classes.

The interval vector for the above set [C, E, G, Bb] is:

<	0	1	2	1	1	1	>
	ic1	ic2	ic3	ic4	ic5	ic6	

What is the interval vector for a major triad [0, 4, 7]? For a major scale?

11. **Normal Order** The set we've been working with, [0, 4, 7, 10], forms a major-minor seventh chord. Remembering that the choice of a particular pitch class as "0" is arbitrary, we can make any pitch class in our set equal to 0. Doing this generates the following sets:

with C = 0	[0, 4, 7, 10]
with E = 0	[0, 3, 6, 8]
with G = 0	[0, 3, 5, 9]
with Bb = 0	[0, 2, 6, 9]

All of the above sets are *permutations,* or re-orderings, of the original set. In tonal theory we would call these chords "1st inversion," "2nd inversion," and "3rd inversion." Inversion has a different meaning in set theory, though, so we call these sets *permutations* or *rotations*.

With so many possible permutations of the same pc set, how do we decide which label to use? *Normal order* is the conventional way of labeling pc sets, and is the "most compact" permutation of a set. Here's the long way to find the normal order (we'll learn shortcuts later):

1. Write down all of a pc set's permutations.
2. Find the permutation with the smallest outer interval.
3. In case of a tie, find the set which is most closely "packed to the left."

In the above example, the smallest outer interval is 8 (minor 6th); so [0, 3, 6, 8] is the normal order for a dominant seventh chord.

<u>Shortcut</u>: play the chord on the piano or write it on staff paper, using each pitch class in turn as the bottom note; find out which permutation has the smallest range from bottom to top.

12. **Inversion** Consider this major triad:

		7		0		
	4	G	If we turn it upside	C	8	
0	E		down, we get:		Ab	5
C						F

To find the normal order of [5, 8, 0], subtract 5 to get [0, 3,7].

Thus, the minor triad is the inversion of the major triad. In the example above, we are thinking of the pc set as a succession of intervals. In fact, *any* pc set can be expressed as a succession of intervals. Here's another example:

Original set:	[0, 1, 3, 5, 7]	
	V V V V	
intervals:	1 2 2 2	now reverse the interval succession:
	2 2 2 1	then rewrite the pc set starting with 0:
	/\ /\ /\ /\	
	[0, 2, 4, 6, 7]	this is the inversion of the original set, written in normal order.

WARNING: Do not reduce intervals to <u>interval class</u> when making inversions. If you encounter an interval greater than 6 half steps, leave it the way it is.

It is easy to check your work when inverting a pc set: the first and last pc should always be the same for both the original and its inversion.

<u>Shortcut</u>: If you have played the set on the piano (or written it on staff paper), simply count half-steps from the top note going down. In other words, you're counting the set "upside down" instead of "right-side up."

13. **Prime Form** In set theory, a pc set and its inversion are considered to be equivalent. This means that in set theory, a major triad is equivalent to a minor triad! This seems rather strange, but when you consider that major and minor triads sound quite a bit more alike than other trichords, say, [0, 1, 4], it starts to make sense.

Prime form is simply the "most normal order" of a pc set. To find the prime form of the major triad, we compare its normal order [0, 4, 7] with its inversion [0, 3, 7]. The latter is the "most normal" (most compact), so the prime form for any major or minor triad is [0, 3, 7].

Example: Find the prime form for the dominant seventh chord.

1. Find its normal order. We already know from above that its normal order is [0, 3, 6, 8].

2. Invert the normal order. This gives us [0, 2, 5, 8].

3. Since the inversion is "more normal" than the original, [0, 2, 5, 8] is the prime form. (How is this chord labeled in tonal theory?)

Remember that in case of a tie in an outer interval, you must find the set most closely packed to the left.

(a) Consider the set [0, 3, 4, 6, 9]. The rotations are

[0, 3, 4, 6, 9]
[0, 1, 3, 6, 9]
[0, 2, 4, 8, 11]
[0, 3, 6, 9, 10]
[0, 3, 6, 7, 9]

In three different rotations of this set, the last element (outer interval) is 9. Since it's a three-way tie, we must examine all three of these sets that end in 9, along with their inversions. This yields the following:

first set:	[0, 3, 4, 6, 9]
inversion:	[0, 3, 5, 6, 9]
second set:	[0, 1, 3, 6, 9]
inversion:	[0, 3, 6, 8, 9]
third set:	[0, 3, 6, 7, 9]
inversion:	[0, 2, 3, 6, 9]

The second set, starting with [0, 1], is the most closely packed of all of these sets, so it is the prime form. Unfortunately, many sets have 3-way ties similar to this one, and require extra work to find their prime forms.

(b) Consider the set [0, 1, 7, 8]. The rotations are

[0, 1, 7, 8]
[0, 6, 7, 11]
[0, 1, 5, 6]
[0, 4, 5, 11]

Obviously, [0, 1, 5, 6] is the rotation with the smallest outer interval. But when we invert [0, 1, 5, 6], we get [0, 1, 5, 6]. This is the prime form of a *symmetrical set*. There are many other symmetrical sets, including the diminished seventh chord and augmented triad.

14. Lists of sets, published in John Rahn's *Basic Atonal Theory* and Joseph Straus' *Introduction to Post-Tonal Theory* among other texts, show all possible pc sets with cardinality 3 through 9. As above, prime forms are shown with [ ], and interval vectors are shown with < >. There is also a complete list of sets at www.stephenandrewtaylor.net/setfinder.

(a) Sets are listed only in *prime form*. If you can't find the set you are looking for, then you don't have the set in prime form. There are no errors on the list. (But there is a subtle distinction between the Forte prime form and the Rahn prime form. If you're interested, check out John Rahn's book *Basic Atonal Theory*.)

(b) The hyphenated numbers are *Forte numbers*, named after Yale theorist Allen Forte, who introduced the numbers in his book *The Structure of Atonal Music*. The Forte number has two parts: the first part is the cardinality of the set; the second part is Forte's catalog number for that set. For example, set 4-2 is Forte's second set of cardinality 4.

(c) When the letter 'z' appears in the name of the set, that set is a member of a 'z' related pair. Z-related sets have identical interval vectors, but one set cannot be derived from the other set through rotation or inversion.